GHENT UNIVERSITY

FACULTY OF ENGINEERING AND ARCHITECTURE

ℓ_0 and ℓ_1 approaches to sparse coding of octonion valued signals

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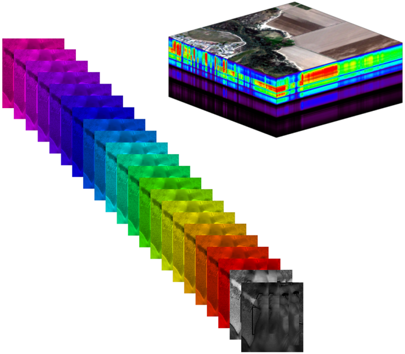
Outline

1. Motivation
2. Octonion based sparse coding
   - Octonion based signal processing
   - Sparse coding
3. Experimental results
4. Conclusion

Outline

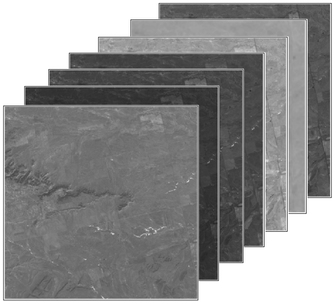
Highdimensional data



Remote-sensing data
(hyperspectral, multispectral, visible. . .)

Multimodal images
(infrared, X-ray, visible . . .)

Motivation - multichannel data



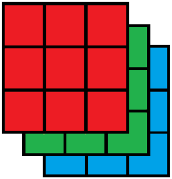
LANDSAT 7 multispectral image

Contrast adjusted RGB image

Outline

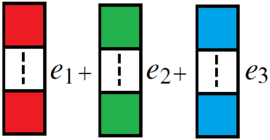
Outline

Color image models



Three color channels

Concatenated model

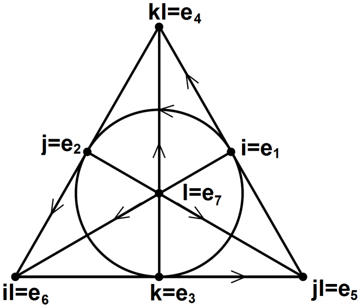


Quaternion based model

Octonions

- The octonion algebra $\mathbb{O} := \mathbb{H} \oplus \mathbb{H}\ell$
- Every element $\dot{a} \in \mathbb{O}$ can be written as

$$\dot{a} = a_0 + a_1 e_1 + a_2 e_2 + \cdots + a_7 e_7 = \sum_{i=0}^{7} a_7 e_7$$



Octonionic multiplication

Octonionic multiplication

For two octonions

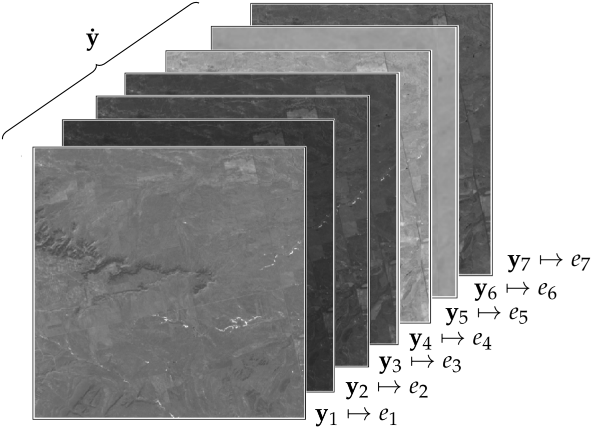
$$\dot{a} = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$$

and

$$\dot{b} = b_0 + b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + b_7e_7$$

we obtain

$$\begin{aligned}
\text{vec}(\dot{a}\dot{b}) &= \begin{bmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1 & a_0 & -a_3 & a_2 & -a_5 & a_4 & a_7 & -a_6 \\ a_2 & a_3 & a_0 & -a_1 & -a_6 & -a_7 & a_4 & a_5 \\ a_3 & -a_2 & a_1 & a_0 & -a_7 & a_6 & -a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & -a_1 & -a_2 & -a_3 \\ a_5 & -a_4 & a_7 & -a_6 & a_1 & a_0 & a_3 & -a_2 \\ a_6 & -a_7 & -a_4 & a_5 & a_2 & -a_3 & a_0 & a_1 \\ a_7 & a_6 & -a_5 & -a_4 & a_3 & a_2 & -a_1 & a_0 \end{bmatrix} \cdot \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix} \\
&= L_o(\dot{a})\text{vec}(\dot{b})
\end{aligned}$$

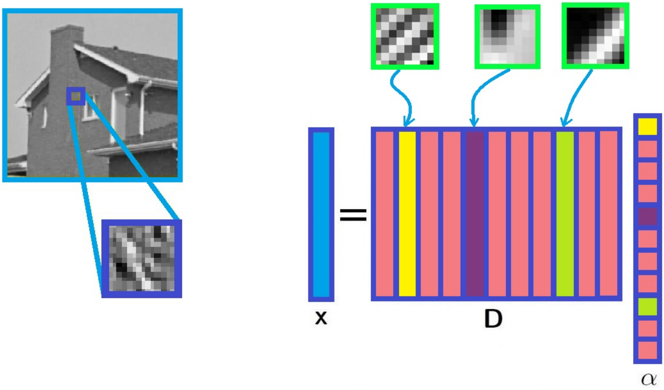


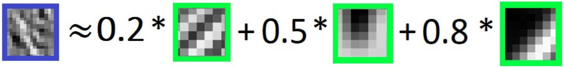
$$\dot{\mathbf{y}} = 0 + \mathbf{y}_1 e_1 + \cdots + \mathbf{y}_7 e_7, \quad \dot{\mathbf{y}} \in \mathbb{O}^{m\times 1}, \ \mathbf{y}_i \in \mathbb{R}^{m\times 1}$$

Landsat 7 image patch as an octonion vector $\dot{\mathbf{y}}$

Outline

Sparse representation





$$\approx 0.2 * d_{200} + 0.5 * d_{124} + 0.8 * d_{59}$$

Octonion problem formulation

Octonion sparse representation

Let $\dot{\mathbf{x}} \in \mathbb{O}^{m\times 1}$ be an $\sqrt{m} \times \sqrt{m}$ image patch, then it can be represented as

$$\dot{\mathbf{x}} = 0 + \mathbf{x}_1 e_1 + \cdots + \mathbf{x}_7 e_7, \quad \mathbf{x}_i \in \mathbb{R}^{m\times 1}.$$

The goal: for a given dictionary $\dot{\mathbf{D}} = \mathbf{D}_0 + \mathbf{D}_1 e_1 + \cdots + \mathbf{D}_7 e_7 \in \mathbb{O}^{m\times n}$ find a sparse code $\dot{\alpha} = \alpha_0 + \alpha_1 e_1 + \cdots + \alpha_7 e_7 \in \mathbb{O}^{n\times 1}$ such that

$$\dot{\mathbf{x}} \approx \dot{\mathbf{D}}\dot{\alpha}.$$

The coefficient matrix is obtained as

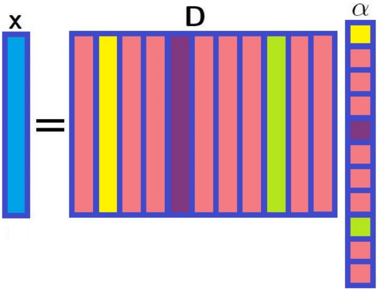
$$\mathbf{C_O} = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 & -\alpha_5 & \alpha_4 & \alpha_7 & -\alpha_6 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 & -\alpha_6 & -\alpha_7 & \alpha_4 & \alpha_5 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 & -\alpha_7 & \alpha_6 & -\alpha_5 & \alpha_4 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_5 & -\alpha_4 & \alpha_7 & -\alpha_6 & \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_6 & -\alpha_7 & -\alpha_4 & \alpha_5 & \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_6 & -\alpha_5 & -\alpha_4 & \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix}$$

Octonion problem formulation

Octonion sparse coding problem

For a given dictionary $\dot{\mathbf{D}} = \{\dot{\mathbf{d}}_k\}_{k=1}^{m} \in \mathbb{O}^{m\times n}$ and signal $\dot{\mathbf{x}} \in \mathbb{O}^{m\times 1}$ solve

$$\hat{\dot{\alpha}} = \arg\min_{\dot{\alpha}} \|\dot{\mathbf{x}} - \dot{\mathbf{D}}\dot{\alpha}\|_2^2 \quad \text{s.t.} \quad \|\dot{\alpha}\|_0 \leq L.$$



S. Lazendić, H. De Bie and A. Pižurica,
Octonion Sparse Representation for Color and Multispectral Image Processing,
Proceedings of the 26th European Signal Processing Conference (EUSIPCO 2018), Rome, Italy, 2018.

Octonion sparse coding

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Idea: OMP over $\mathbb{O}$

Difficulty: $\arg\min_{\dot{\alpha}} \|\dot{\mathbf{x}} - \dot{\mathbf{D}}\dot{\alpha}\|_2^2$ **over $\mathbb{O}$**

Octonion sparse coding

Octonion sparse coding problem

For a given dictionary $\dot{\mathbf{D}} = \{\dot{\mathbf{d}}_k\}_{k=1}^{m} \in \mathbb{O}^{m\times n}$ and signal $\dot{\mathbf{x}} \in \mathbb{O}^{m\times 1}$ solve

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Idea: OMP over $\mathbb{O}$

- $\nu : \mathbb{O}^{n\times 1} \to \mathbb{R}^{8n\times 1}$
- $\chi : \mathbb{O}^{m\times n} \to \mathbb{R}^{8m\times 8n}$

s.t.

- $\|\dot{\alpha}\|_2^2 = \|\nu(\dot{\alpha})\|_2^2$
- $\nu(\dot{\mathbf{D}}\dot{\alpha}) = \chi(\dot{\mathbf{D}})\nu(\dot{\alpha})$

Difficulty: $\arg\min_{\dot{\alpha}} \|\dot{\mathbf{x}} - \dot{\mathbf{D}}\dot{\alpha}\|_2^2$ **over** $\mathbb{O}$

$$\begin{aligned} \|\dot{\mathbf{x}} - \dot{\mathbf{D}}\dot{\alpha}\|_2^2 &= \|\nu(\dot{\mathbf{x}} - \dot{\mathbf{D}}\dot{\alpha})\|_2^2 \\ &= \|\nu(\dot{\mathbf{x}}) - \chi(\dot{\mathbf{D}})\nu(\dot{\alpha})\|_2^2 \end{aligned}$$

$$\Longrightarrow \quad \dot{\alpha} = \nu^{-1}\left(\chi(\dot{\mathbf{D}})^{\dagger}\nu(\dot{\mathbf{x}})\right)$$

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Octonion Sparse Representation for Color and Multispectral Image Processing,
Proceedings of the 26th European Signal Processing Conference (EUSIPCO 2018), Rome, Italy, 2018.

Octonion sparse coding problem - ℓ_1 minimization (SOCP approach)

For a given dictionary $\dot{\mathbf{D}} = \{\dot{\mathbf{d}}_k\}_{k=1}^{m} \in \mathbb{O}^{m \times n}$ and signal $\dot{\mathbf{x}} \in \mathbb{O}^{m \times 1}$ solve

$$\hat{\dot{\alpha}} = \arg\min_{\dot{\alpha}} \|\dot{\mathbf{x}} - \dot{\mathbf{D}}\dot{\alpha}\|_2^2 + \lambda \|\dot{\alpha}\|_1.$$

This problem can be equivalently written as

$$\min_{t \in \mathbb{R}^+} t \quad \text{s.t.} \quad \dot{\mathbf{y}} = \dot{\mathbf{D}}\dot{\mathbf{x}}, \quad \|\dot{\mathbf{x}}\|_1 \leq t.$$

By decomposing

$$t = \sum_{i=1}^{n} t_i, \quad t_i \in \mathbb{R}^+$$

we can write the last constraint as

$$\|\dot{\mathbf{x}}\|_1 = \sum_{i=1}^{n} \|\dot{\mathbf{x}}_i\|_2 = \sum_{i=1}^{n} \|v(\dot{\mathbf{x}}_i)\|_2 \leq \mathbf{1}^T\mathbf{t} = \mathbf{1}^T \cdot [t_1, \ldots, t_n] = t.$$

- The ℓ_1 minimization problem then becomes

$$\min_{t \in \mathbb{R}^+} \mathbf{1}^T \mathbf{t} \ \text{ s.t. } \ \dot{\mathbf{y}} = \dot{\mathbf{D}} \dot{\mathbf{x}}, \ \ \|\nu(\dot{\mathbf{x}}_i)\|_2 \leq t_i,$$

for every $i = 1, \ldots, n$.

- Denote for every $i = 1, \ldots, n$

$$\begin{aligned}
\hat{\mathbf{x}} &= [t_1, \nu(\dot{\mathbf{x}}_1), \ldots, t_n, \nu(\dot{\mathbf{x}}_n)]^T \in \mathbb{R}^{9n \times 1}, \\
\hat{\mathbf{c}} &= [c_j]_j = \begin{cases} 1, & \text{if } j = 9i - 8 \\ 0, & \text{otherwise}, \end{cases}, \\
\hat{\mathbf{y}} &= \nu(\dot{\mathbf{y}}) \in \mathbb{R}^{8m \times 1}, \\
\hat{\mathbf{D}} &= \left[\mathbf{0}, \chi\left(\dot{\mathbf{d}}_1\right), \ldots, \mathbf{0}, \chi\left(\dot{\mathbf{d}}_n\right)\right] \in \mathbb{R}^{8m \times 9n}
\end{aligned}$$

- Problem

$$\hat{\dot{\alpha}} = \arg\min_{\dot{\alpha}} \|\dot{\mathbf{x}} - \dot{\mathbf{D}} \dot{\alpha}\|_2^2 + \lambda \|\dot{\alpha}\|_1.$$

can be equivalently written in the form of the *second-order cone optimization problem* as

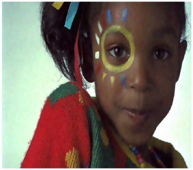
$$\min_{\hat{\mathbf{x}} \in \mathbb{R}^{\mathbf{9n}}} \hat{\mathbf{c}}^T \hat{\mathbf{x}} \ \text{ s.t. } \ \hat{\mathbf{y}} = \hat{\mathbf{D}} \hat{\mathbf{x}}, \ \ \|\nu(\dot{\mathbf{x}}_i)\|_2 \leq t_i.$$

Outline

Color image denoising

(a) $\sigma = 25$

(b) K-SVD 29.20 dB



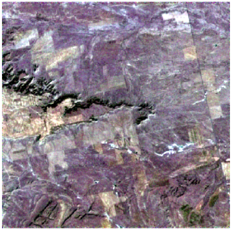
(c) K-QSVD 31.21 dB

(d) ODL_2 34.12 dB

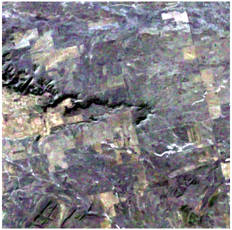
	Average values of PSNR/SSIM		
	K-SVD	K-QSVD	ODL
$\sigma = 10$	34.40dB/0.966	34.53dB/0.902	**35.83dB/0.990**
$\sigma = 25$	29.00dB/0.880	30.65dB/0.819	**32.21**dB/**0.935**

Denoising of color images

(e) $\sigma = 10$



(f) K-SVD



(g) ODL

	Montana image		Mississippi image	
	K-SVD	ODL	K-SVD	ODL
$\sigma = 10$	34.79/**0.991**	**36.79**/0.900	36.21/**0.989**	**38.71**/0.918
$\sigma = 25$	31.63/**0.981**	**32.30**/0.765	32.54/**0.969**	**33.09**/0.772
Average	33.21/**0.986**	**34.54**/0.832	34.37/**0.979**	**35.90**/0.845

Denoising of Landsat 7 data

Outline

Conclusion

- Possible applications in various inverse problems and detection/classification tasks
- Generalization of the real and the quaternion sparse model
- Processing of multichannel images in a holistic manner
- Better preservation of interchannel dependencies
- Theoretical foundation and analysis of different methods

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