# Compressed Sensing Using Sparse Binary Measurements: A Rateless Coding Perspective

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***Abstract*—Compressed Sensing (CS) methods using sparse binary measurement matrices and iterative message-passing recovery procedures have been recently investigated due to their low computational complexity and excellent performance. Drawing much of inspiration from sparse-graph codes such as Low-Density Parity-Check (LDPC) codes, these studies use analytical tools from modern coding theory to analyze CS solutions. In this paper, we consider and systematically analyze the CS setup inspired by a class of efficient, popular and flexible sparse-graph codes called rateless codes. The proposed *rateless* CS setup is asymptotically analyzed using tools such as Density Evolution and EXIT charts and fine-tuned using degree distribution optimization techniques.**

***Keywords*—compressed sensing, rateless codes, iterative decoding, density evolution, EXIT charts.**

## I. Introduction

In recent years, a number of studies considered the interplay between the compressed sensing (CS) and modern coding theory. Sarvotham et al. were the first to consider reconstruction of strictly sparse signals using sparse binary measurement matrices and iterative message-passing reconstruction algorithms [1]. Their scheme, called Sudocodes, is later identified to be a version of verification-based (VB) decoding algorithms (LM1/LM2) previously proposed for the decoding of Low-Density Parity Check (LDPC) codes [2]. The same authors proposed more general framework called Compressed Sensing via Belief Propagation (CSBP), introducing factor graph modeling and Belief-Propagation reconstruction to recover sparse signals from noisy measurements [3]. However, CSBP is rather complex both from the analysis and implementation perspective, since the messages exchanged across factor graph represent continuous probability distributions. As low-complexity versions of CSBP amenable to rigorous analysis, VB decoding algorithms (such as LM1 and LM2) have been investigated in noiseless CS scenario in [4][5]. Using the CS setup analogue to LDPC codes, in these works, coding-theoretic tools such as the density evolution (DE) and stopping set analysis are applied to assess the CS recovery performance. A related class of recovery algorithms that operate as message-passing on sparse graphs, called Interval Propagation Algorithms (IPA), have also been recently considered in the CS context and analyzed using coding-theoretic tools borrowed from LDPC codes analysis [6][7].

In this paper, we consider a CS setup that applies sparse binary measurement matrices and iterative recovery procedures in a way that is inspired by rateless coding theory [8][9]. We aim to explore the benefits of rateless coding: simplicity, efficiency, flexibility and versatile analysis tools, in order to design novel or improve current CS methods. Although connections between CS and rateless codes have been indicated elsewhere (see, e.g., [10]), to the best of our knowledge, no systematic study is currently available. As a starting point, in this study, we provide asymptotic analysis (both using the DE and the EXIT chart approach) of a *rateless* CS system. The key observation here, in contrast to the rateless coding setup, is that the noiseless rateless CS setup considered in this paper inherently exhibits unequal error protection (UEP) property. Namely, the feature of efficient verification-based recovery procedures is to favour recovery of zero-valued signal coefficients along the recovery process. Thus using UEP-based asymptotic analysis derived in this paper, we are able to asymptotically predict both zero and non-zero signal recovery performance as a function of the number of received signal measurements. Furthermore, we are able to address the optimized design of the rateless CS system and fine-tune it using the degree distribution optimization techniques.

The rateless CS setup we consider is not only relevant in terms of the CS methods that use sparse binary measurement matrices and iterative recovery procedures. Rather, we set our system model in the communication-theoretic framework that includes both compressive signal acquisition and rateless data communication, giving a strong flavour of joint source-channel coding (JSCC) CS system [11][12]. Although simplified to noiseless scenario (i.e., equivalent to erasure coding scenario) for the convenience of analysis, the results and ideas presented could be applied and extended to modelling JSCC systems where remote compressive signal acquisition is combined with reliable and adaptive rateless communications across wireless access links. Finally, we note that several recently proposed efficient CS methods based on two-stage approach [1][13][14], effectively solve the same problem that Raptor coding solved in rateless scenario [9]: an analogy that could be exploited for devising new two-stage CS methods.

## II. Background

We consider compressed sensing (CS) signal acquisition where a sparse signal $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ containing small number $K \ll N$ of non-zero elements is recovered from a set of measurements $\mathbf{y} = (y_1, y_2, \ldots, y_M) \in \mathbb{R}^M$ where $K < M \ll N$. The measurement set $\mathbf{y} = \mathbf{\Phi} \cdot \mathbf{x}$ is obtained as a sequence of signal projections onto the row-vectors $\phi_i, 1 \leq i \leq M$, of so called measurement matrix $\mathbf{\Phi}$. We consider sparse binary row-vectors $\phi_i$, thus each measurement $y_i$ is a sum of a small subset of signal elements.

We define a measurement graph $\mathcal{G} = (\mathcal{V} = \mathcal{S} \cup \mathcal{M}, \mathcal{E})$ as a bipartite graph consisting of the set $\mathcal{S}$ of $N$ signal nodes (SNs) that correspond to signal elements $x_1, x_2, \ldots, x_N$, and the set $\mathcal{M}$ of $M$ measurement nodes (MNs) that correspond to a sequence of measurements $y_1, y_2, \ldots, y_M$. The set $\mathcal{E}$ is in one-to-one correspondence with non-zero entries of $\mathbf{\Phi}$, i.e., an edge $e = (x_j, y_i) \in \mathcal{E}$ of the graph connects a MN $y_i$ with a SN $x_j$ iff $\phi_i(j) = 1$. The neighbour set of MN $y_i$: $\mathcal{N}(y_i) = \{x_j : (x_j, y_i) \in \mathcal{E}\}$, contains all the SNs that participate in the measurement $y_i$. The degree $d_i = |\mathcal{N}(y_i)|$ of the MN $y_i$ is the number of edges incident to $y_i$. The graph $\mathcal{G}$ may be described using SN and MN degree distributions $\Lambda(x) = \sum_{i=1}^{d^{(s)}_{max}} \Lambda_i \cdot x^i$ and $\Omega(x) = \sum_{i=1}^{d^{(m)}_{max}} \Omega_i \cdot x^i$, where $\Lambda_i$ and $\Omega_i$ are the fraction of SNs and MNs of degree $i$, respectively, while $d^{(s)}_{max}$ and $d^{(m)}_{max}$ are maximum SN and MN degrees. Besides the above node-oriented degree distributions, it is useful to introduce edge-oriented degree distributions[1] $\lambda(x) = \sum_{i=1}^{d^{(s)}_{max}} \lambda_i \cdot x^{i-1}$ and $\omega(x) = \sum_{i=1}^{d^{(m)}_{max}} \omega_i \cdot x^{i-1}$.

The sparse structure of $\mathcal{G}$ allows for low-complexity signal reconstruction using iterative message-passing algorithms previously applied in decoding of sparse-graph codes. This inspired the authors of Sudocodes scheme to propose such reconstruction methods [1], later identified to be instances of LM1 and LM2 decoding algorithms for Low-Density Parity Check (LDPC) codes [2]. For completeness of exposition, the two algorithms are repeated below.

**LM1 Reconstruction**: Let $\mathcal{G}(t)$ be the measurement graph after $t$ iterations of reconstruction procedure. Given MN values $\mathbf{y}$ and the initial measurement graph $\mathcal{G}(0)$, the LM1 operates iteratively over $\mathcal{G}$ by applying the following rules:

**LM1 Reconstruction**
**while** $(\mathcal{G}(t) \neq \emptyset) \wedge (\mathcal{G}(t) \neq \mathcal{G}(t-1))$
**begin**
[process sequentially all MNs of $\mathcal{G}(t)$]
**Zero-MN Rule:**
**if** $(y_i = 0)$
**begin**
$\forall x_j \in \mathcal{N}(y_i) : x_j := 0;$
$\mathcal{E} := \mathcal{E} \setminus (x_j, y_i); \mathcal{V} := \mathcal{V} \setminus \{\mathcal{N}(y_i), y_i\};$
**end**
**Singleton-MN Rule:**
**if** $(y_i \neq 0) \wedge (|N(y_i)| = 1)$
**begin**
$\forall x_j \in \mathcal{N}(y_i) : x_j := y_i;$
$\mathcal{E} := \mathcal{E} \setminus (x_j, y_i); \mathcal{V} := \mathcal{V} \setminus \{y_i\};$
$\forall y_k \in \mathcal{N}(x_j) : y_k := y_k - x_j;$
$\mathcal{E} := \mathcal{E} \setminus (x_j, y_k); \mathcal{V} := \mathcal{V} \setminus \{x_j\};$
**end**
**end**

**LM2 Reconstruction**: In addition to LM1 rules, the LM2 procedure adds an additional rule as described next:

**LM2 Reconstruction**
**Equal-MN Rule:**
[for each MN $y_i$ - process all MNs $y_j$ where $j > i$]
**if** $(\mathcal{N}(y_i) \cap \mathcal{N}(y_j) = \{x_k\}) \wedge (y_i = y_j)$
**begin**
$x_k := y_i := y_j;$
$\forall x_l \in \{\mathcal{N}(y_i) \cup \mathcal{N}(y_j) \setminus x_k\} : x_l := 0;$
$\forall x_e \in \{\mathcal{N}(y_i)\} : \mathcal{E} := \mathcal{E} \setminus (x_e, y_i);$
$\forall x_e \in \{\mathcal{N}(y_j)\} : \mathcal{E} := \mathcal{E} \setminus (x_e, y_j);$
$\mathcal{V} := \mathcal{V} \setminus \{\mathcal{N}(y_i), \mathcal{N}(y_j), y_i, y_j\};$
**end**

## III. System Model and Rateless Coding Analogy

We set up the *rateless* CS system model from communications theory perspective. It is comprised of: i) the measurement system, ii) the communication channel, and iii) the reconstruction system (Fig. 1) We next discuss each of the components.

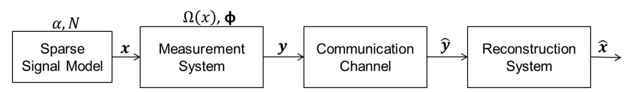


Fig. 1. Rateless CS system model.

**Measurement System:** we assume reconfigurable measurement system that sequentially samples the signal $\mathbf{x}$ using a sequence of sparse binary measurement row-vectors $\phi_i, i \geq 1$, in order to produce a stream of measurements $\mathbf{y}$. The number of measurements $M$ may be arbitrarily extended in a rateless fashion. The measurement $\phi_i$ is defined by the degree $d_i$; unless otherwise stated, the $d_i$ neighbours of MN $y_i$ are selected uniformly at random among SNs $\mathbf{x}$. The measurement process is defined using the MN degree distribution $\Omega(x)$ which is used to draw the sequence of degree's $d_i$. For simplicity, the measurement process is assumed noiseless.

**Communication Channel**: the stream of measurements $\mathbf{y}$ are sequentially communicated across a communication channel to provide an input stream $\hat{\mathbf{y}}$ to the reconstruction system. For the initial study, we consider idealized case where the noise-free channel conveys the real numbers without errors/erasures, i.e., we assume $\hat{\mathbf{y}} = \mathbf{y}$. We leave consideration of realistic channel models, including noise, erasures, quantization and specific packetization models, for future work.

**Reconstruction System**: we consider CS reconstruction using low-complexity iterative message-passing recovery algorithms. In particular, we assume verification-based reconstruction (LM1 and LM2 algorithms) is implemented at the receiver [2][4]. The reconstruction is performed progressively with the arrival of each new measurement until the signal is fully recovered, which is signalled back to the measurement system by a feedback message.

*Rateless Coding Perspective*: As described above, the measurement system is equivalent to rateless (more precisely, LT) encoder. In case the acquired signal satisfies $\mathbf{x} \in \mathbb{F}_{2^q}^N$ and is not sparse, the above scenario reduces to rateless coding problem. In that case, the reconstruction system that applies LM1 recovery using only Singleton-MN rule is equivalent to

[1] They are obtained as $\lambda(x) = \Lambda'(x)/\Lambda'(1)$ and $\omega(x) = \Omega'(x)/\Omega'(1)$, where $\Lambda'(x) = d\Lambda(x)/dx$ and $\Omega'(x) = d\Omega(x)/dx$.

the standard iterative rateless decoder [8]. Thus rateless coding system is a special case of the considered rateless CS model.

Measurement system that connects each MN to randomly uniformly sampled subset of SNs operates akin to equal error protection (EEP) rateless codes. The key observation that underlies most of the analysis in the rest of the paper is that, for the CS setup above, even if the EEP measurement process is used, it is the reconstruction system that introduces unequal error protection (UEP) effect among classes of zero and non-zero SNs. In particular, for LM1 recovery, the Zero-MN rule boosts recovery of zero-valued SNs, and for LM2 recovery, Equal-MN rule further enhances the UEP effect. In the following, inspired by analysis of UEP rateless codes [15][16], we analyze rateless CS system performance using LM1/LM2 reconstruction algorithms.

It is worth noting that, inspired by the weighted [15] and the windowed [16] UEP rateless codes, recently, non-uniform CS schemes are proposed [10][17]. However, the efficiency of these schemes varies depending on whether the measurement system possess prior knowledge (or learning capabilities) on where to search for more important SNs.

## IV. Asymptotic Analysis

We apply standard asymptotic analysis tools for sparse-graph codes to analyze the above rateless CS system. For simplicity and insight, we consider noiseless measurement system and ideal channel model - we note that including erasures of measurements in the channel could be straightforwardly accommodated in our results.

### A. Density Evolution Analysis

A special case of density evolution (DE) analysis called AND-OR tree analysis is usually applied to asymptotically evaluate the performance of rateless erasure codes [18]. AND-OR tree analysis can be reshaped for the CS setup and LM1/LM2 recovery. The key observation is that recovery probabilities of zero and non-zero valued SNs behave differently (UEP effect) thus leading to two-dimensional coupled system of recursive equations as described below.

*Lemma 4.1:* **[LM1 reconstruction asymptotic analysis]** Let $\mathcal{G}$ be described by SN and MN degree distributions $\Lambda(x)$ and $\Omega(x)$. Let $p_l^{(z)}$ and $p_l^{(nz)}$ denote the asymptotic probabilities that zero and non-zero SNs are not recovered after $l$ iterations of the LM1 reconstruction, respectively. Then:

$$p_l^{(nz)} = \lambda\Big(1 - \sum_{i=1}^{d_{max}^{(m)}} \omega_i \cdot \sum_{j=0}^{i-1} \binom{i-1}{j} (\alpha\bar{p}_{l-1}^{(nz)})^j \cdot (\bar{\alpha}\bar{p}_{l-1}^{(z)})^{i-1-j}\Big) \tag{1}$$

$$p_l^{(z)} = \lambda\Big(1 - \sum_{i=1}^{d_{max}^{(m)}} \omega_i \cdot \sum_{j=0}^{i-1} \binom{i-1}{j} (\alpha\bar{p}_{l-1}^{(nz)})^j \cdot \bar{\alpha}^{i-1-j}\Big), \tag{2}$$

where $\alpha = K/N$ is called sparsity factor; and we use compact notation $\bar{x} = 1 - x$ (where $x$ is constant, variable or function evaluation). The initial values are $p_0^{(z)} = p_0^{(nz)} = 1$.

*Proof:* The proof follows from the fact that, for MN to recover a non-zero valued SN, it needs to recover all remaining

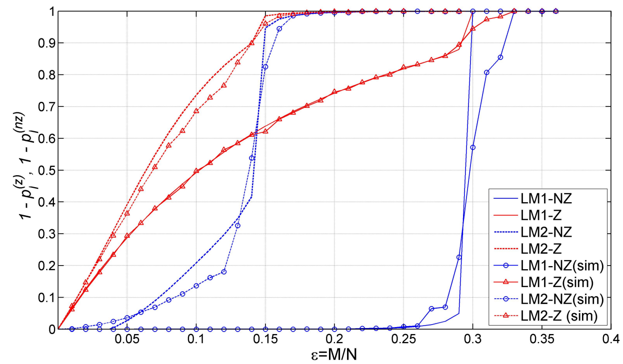


Fig. 2. Asymptotic ($N, l \to \infty$) and simulated ($N = 5000$) performance of rateless CS as a function of $\epsilon = M/N$ for LM1/LM2 recovery (Z and NZ denotes zero and non-zero coefficients, respectively).

zero and non-zero SN neighbours. In contrast, to recover zero-valued SN, MN needs to know only its non-zero valued SN neighbours, after which its value drops to zero thus revealing remaining zero-valued SN neighbours.

In CS, similarly to rateless coding scenario, the measurement subsystem can control only $\Omega(x)$, while $\Lambda(x)$ depends on how SNs are sampled by MNs. If this selection is uniformly at random, then $\Lambda(x)$ asymptotically tends to the Poisson distribution, and $\lambda(x) = e^{\mu\epsilon(x-1)}$. In the latter, $\mu = \Omega'(1)$ is the average measurement node degree, and $\epsilon = M/N$ is known as the reception overhead in rateless coding, while in CS, $\epsilon$ is the product of the oversampling ratio $\theta = M/K$ and the sparsity factor $\alpha = K/N$.

*Lemma 4.2:* **[LM2 reconstruction asymptotic analysis]** The asymptotic non-recovery probabilities $p_l^{(z)}$ and $p_l^{(nz)}$ of LM2 recovery have the same form as equations (1) and (2), except that in each equation, one of the terms within the inner sum is updated. In particular, the terms $j = 0$ in eq. (1) and $j = 1$ in eq. (2), are respectively replaced by:

$$\bar{\alpha}^{i-1}\Big((\bar{p}_{l-1}^{(z)})^{i-1} + (1 - (\bar{p}_{l-1}^{(z)})^{i-1}) \cdot (1 - \frac{\lambda(\bar{\omega}(\bar{\alpha}))}{\bar{\omega}(\bar{\alpha})})\Big) \tag{3}$$

$$(i-1) \cdot \bar{\alpha}^{i-2} \cdot \alpha\Big(\bar{p}_{l-1}^{(nz)} + p_{l-1}^{(nz)} \cdot \bar{\lambda}(\bar{\omega}(\bar{\alpha}))\Big). \tag{4}$$

*Proof:* The proof of LM2 case, as for LM1 case, follows from detailed analysis of AND-OR tree recovery. For LM2 case, it is somewhat more involved. The details of both proofs are relegated to the extended version of this paper.

*Example 4.1:* Fig. 2 shows asymptotic recovery probabilities (as $N \to \infty$) obtained from *Lemma 1* and *Lemma 2* for MN-regular CS scheme where $\Omega(x) = x^{30}$ after LM1 and LM2 reconstruction of input message of sparsity-factor $\alpha = 0.05$ for zero and non-zero input symbols. The same figure illustrates finite-length simulated performance of both schemes for signal length $N = 5000$. We note excellent agreement between simulation results for large but finite-length signals and the predicted asymptotic behaviour.

### B. EXIT-Chart Analysis

Another useful method to asymptotically analyze sparse-graph recovery process applies extrinsic information transfer

(EXIT) charts [19]. For erasure coding scenario, EXIT charts are known to be equivalent to AND-OR tree analysis. For completeness, and due to the fact that EXIT charts are useful independently of AND-OR tree analysis, we reshape EXIT-chart analysis for CS scenario and LM1/LM2 recovery. Due to space constraints, we provide EXIT-chart description only for LM1 recovery (the LM2-case follows the same approach).

In the context of CS, EXIT charts track exchange of extrinsic mutual information among two constituent parts of the measurement graph: the MN-part and the SN-part. However, due to LM1/LM2 recovery rules that favour zero-valued SNs, the extrinsic mutual information of zero and non-zero valued SNs behave differently (UEP effect). This leads to two EXIT-chart pairs as functions of two input (mutual information) variables, one pair for both MN-part and SN-part processing.

**LM1 reconstruction - EXIT functions, MN-part:** For CS scenario and LM1 recovery, EXIT functions of the MN-part processing are given by the following coupled pair of functions:

$$I_{E,MN}^{(nz)} = \sum_{i=1}^{d_{max}^{(m)}} \omega_i \cdot \sum_{j=0}^{i-1} \binom{i-1}{j} (\alpha I_{A,MN}^{(nz)})^j \cdot (\bar{\alpha} I_{A,MN}^{(z)})^{i-1-j} \tag{5}$$

$$I_{E,MN}^{(z)} = \sum_{i=1}^{d_{max}^{(m)}} \omega_i \cdot \sum_{j=0}^{i-1} \binom{i-1}{j} (\alpha I_{A,MN}^{(nz)})^j \cdot \bar{\alpha}^{i-1-j}. \tag{6}$$

The above EXIT functions map a pair of input (a priori) mutual information variables $(I_{A,MN}^{(nz)}, I_{A,MN}^{(z)})$ into a pair of output (a posteriori) mutual information variables $(I_{E,MN}^{(nz)}, I_{E,MN}^{(z)})$.

**LM1 reconstruction - EXIT functions, SN-part:** In contrast, the two EXIT functions for the SN-part are essentially a single function applied independently on both input variables:

$$I_{E,SN} = \sum_i \lambda_i \cdot (1 - (1 - I_{A,SN})^{i-1}). \tag{7}$$

Thus the above EXIT function maps $I_{A,SN}^{(nz)}$ and $I_{A,SN}^{(z)}$ into $I_{E,SN}^{(nz)}$ and $I_{E,SN}^{(z)}$, respectively. In usual case, where MNs sample SNs uniformly at random, the right hand side of eq. (7) can be reduced to $1 - e^{-\mu\epsilon I_{A,SN}}$.

*Example 4.2:* We generate a sequence of EXIT function mappings for the CS scheme $\Omega(x) = x^{30}$ after LM1 recovery at the reception overhead $\epsilon = M/N = 0.2912$ of the sparse input message of sparsity-factor $\alpha = 0.05$. The process is initiated at the point $(I_{A,MN}^{(\bar{z})}, I_{A,MN}^{(z)}) = (0, 0)$ and proceeds by iterative exchange of mutual information among MN and SN-part until convergence[2]. The *path of convergence* (see footnote below) ends at the pair $(I_{E,SN}^{(nz)}, I_{E,SN}^{(z)}) = (0.9998, 0.9998)$ in Fig. 3, which, due to equivalence with DE analysis for erasure channel, corresponds exactly to the values of LM1 curves (at $\epsilon = 0.2912$) on Fig. 2.

[2]In usual EXIT charts (e.g., in EEP channel coding), the convergence process is represented by a pair of single-variable EXIT functions: $I_{E,MN}(I_{A,MN})$ and $I_{E,SN}^{-1}(I_{A,SN})$. The extrinsic mutual information exchange either converges to their intersection point (if such exists), or to maximum achievable mutual information if $I_{E,MN}(I_{A,MN}) > I_{E,SN}^{-1}(I_{A,SN})$ for all $I_A \in [0, 1]$; i.e., if the "tunnel" between the curves is open. In CS scenario, due to inherent UEP nature of zero and non-zero SN recovery, we need 3D-EXIT charts. In this case, convergence depends on the existence of a "tunnel" among four *surfaces* over $(I_A^{(nz)}, I_A^{(z)}) \in [0, 1]^2$. Due to the space constraints, we do not visualise 3D-EXIT charts here, but only provide 2D-projection of evolution of $(I_{E,SN}^{(nz)}, I_{E,SN}^{(z)})$ pairs in Fig. 3.

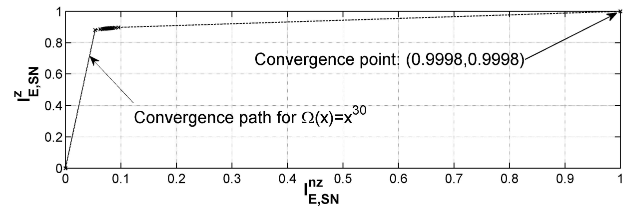


Fig. 3. Evolution of EXIT chart pairs $(I_{E,SN}^{(nz)}, I_{E,SN}^{(z)})$ until convergence.

## V. Degree Distribution Optimization

Given the analytic description of the asymptotic behaviour, the typical next step in the design of rateless coding systems is the optimization of the degree distribution $\Omega(x)$. We proceed the same path for the CS scenario. Using EXIT charts, it was recently shown (see [20], Chap. 5) that the optimization of $\Omega(x)$ (or $\omega(x)$) can be cast as a convex optimization problem. We apply the same approach on the CS setup.

We focus on the standard case where a MN selects neighbouring SNs uniformly at random. The degree distribution of SNs converges to $\lambda(x) = e^{\mu\epsilon(x-1)}$. Denoting $\eta = \Lambda'(1)$ as the average SN degree, we have that $\eta = \mu\epsilon$ depends on the employed MN degree distribution (since $\mu = \Omega'(1)$) and the reception overhead $\epsilon$. For fixed $\eta$, the recovery probability of both zero and non-zero SNs is upper bounded by $e^{-\eta}$, due to SNs that remain uncovered. Finally, by fixing $\eta$ (acceptable error floor), one can search for $\Omega(x)$ that minimizes the overhead conditioned that the recovery probability of both zero and non-zero SNs converges to $1-e^{-\eta}$. In other words, among all $\Omega(x)$ for which EXIT charts converge (i.e., there exists an open "tunnel"), we search for the one which results in the minimal overhead $\epsilon \sim \sum_{i=1}^{d_{max}^{(m)}} \frac{\omega_i}{i}$, as formulated below.

**Degree Distribution Optimization Problem:**

$$\min \quad \sum_{i=1}^{d_{max}^{(m)}} \frac{\omega_i}{i} \tag{8}$$

$$s.t. \quad \omega_i \geq 0 \tag{9}$$

$$\sum_i \omega_i = 1 \tag{10}$$

$$I_{E,MN}^{(nz)}(I_A^{(nz)}, I_A^{(z)}) \geq I_{E,SN}^{-1}(I_A^{(nz)}) \tag{11}$$

$$I_{E,MN}^{(z)}(I_A^{(nz)}) \geq I_{E,SN}^{-1}(I_A^{(z)}) \tag{12}$$

Unfortunately, unlike optimizing standard EEP rateless codes (see [20] and references therein), the above optimization problem is much more involved due to constraints (11) and (12). Namely, once we fix $\Omega(x)$, the convergence path consisting of $(I_{E,SN}^{(nz)}, I_{E,SN}^{(z)})$ points becomes fixed (Fig. 3). Thus the "tunnel" between 3D-EXIT surfaces need not to exist over the entire domain $[0, 1]^2$, but only over the convergence path corresponding to a given $\Omega(x)$. This dependence of the constraints (11) and (12) on the optimization variable $\Omega(x)$ (or $\omega(x)$) is what makes the problem particularly challenging.

TABLE I. OPTIMIZED LM1/LM2 DEGREE DISTRIBUTIONS

| **Recovery method** | **Optimized** $\omega(x)$ ($d_{max}^{(m)} = 30$) | **Reception overhead** $\epsilon$ | **Avg. SN degree** $\eta$ |
|---|---|---|---|
| LM1 | $0.5738x^{20} + 0.0685x^{27} + \ldots$ $\ldots 0.1421x^{28} + 0.2156x^{29}$ | 0.2443 | 6 |
| LM2 | $0.3511x^{20} + 0.1054x^{27} + \ldots$ $\ldots 0.2163x^{28} + 0.3272x^{29}$ | 0.1770 | 4.675 |

Using the insight from constraints (9)-(12), we were able to obtain optimized degree distributions by performing random sequential search over the probability simplex (Table I).

## VI. DISCUSSION AND FUTURE WORK

As a conclusion, we comment on the rateless CS system features that we find promising for future research.

### *A. Two-Stage Reconstruction in Rateless CS*

In rateless coding, it is well known that sampling SNs uniformly at random using the degree distribution $\Omega(x)$ of constant average degree $\eta$ yields an error floor due to uncovered SNs [8][9]. The same problem is noted in CS schemes that apply sparse binary measurement matrices and resolved using two-stage reconstruction methods [1][13][14]. Therein, after the first stage (that employs sparse-graph approaches and iterative recovery methods) recovers majority of SNs, the error floor is cleaned by switching to more powerful but more complex measurement/recovery methods (e.g., a combination of non-sparse measurements and matrix inversion [1], or basis-pursuit [13], or iterative hard-thresholding [14]). This *post-coding* approach is in sharp contrast to how this problem is solved in rateless coding, where *precoding* approach is used [9]. With precoding in the CS scenario, the measurement system could initially produce a fraction of additional SNs, called intermediate SNs, from the original ones. Then, rateless CS approach (as described earlier) could be applied over both original and intermediate SNs and the resulting Raptor-like (bi-layer) graph could be decoded using the iterative methods.

### *B. Adaptivity Feature of Rateless CS*

In contrast to the rateless coding encoder, the CS measurement system may have significant abilities to learn the signal along measurement process. As a simple example, every zero-valued MN reveals its zero-valued SN neighbours that are no longer relevant in subsequent measurements. Some initial steps towards the design of adaptive CS using sparse binary measurements was part of our prior work [21], however, we believe that this topic deserves further analysis.

### *C. Noise-Resistance Feature of Rateless CS*

Majority of the real-world applications of the rateless CS system would involve additive noise, either at the measurement subsystem or within the communication channel, or both. For simplicity and convenience of analysis, our work in this paper is based on the noiseless assumption. This is consistent with the nature of recovery process since LM1/LM2 methods cannot cope with noisy coefficients [4][5]. Noisy extensions have been considered elsewhere [3][14], however, we believe that there is a room for improvement in their simplicity and efficiency. As a promising path in this direction, we identify adaptation of approximate message passing (AMP) [22], originally proposed for high-density graphs, to the rateless CS model.

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